η = y/R
λ_i = eigenvalues of L
λ_i^0 = eigenvalues of L when $\epsilon_i = 0$
λ_i^* = Rayleigh-Ritz estimate of λ_i
Λ_i = eigenfunctions of L
Λ_i^0 = eigenfunctions of L when $\epsilon_i = 0$
$\mathbf{\Lambda}_i^*$ = Rayleigh-Ritz estimate of Λ_i
τ = $t\, \mathcal{D}_{im}/R^2$
χ = x/R
ω_i = approximation functions as defined by Equation (64)

LITERATURE CITED

Gill, W. N., and R. Sankarasubramanian, "Exact Analysis of Unsteady Convection Diffusion," *Proc. Royal Soc.*, **A316**, 341 (1970).

Lee, H.-L., and E. N. Lightfoot, "Preliminary Report on Ultrafiltration-Induced Polarization Chromatography—An Analog of Field-Flow Fractionation," *Sep. Sci.*, **11**, No. 5, 417 (1976).

Reis, J. F. G., and E. N. Lightfoot, "Electropolarization Chromatography," *AIChE J.*, **22**, No. 4, 779 (1976).

Reis, J. F. G., "Separation Processes Based on Electromigration for Fractionation of Proteins," Ph.D. dissertation, Univ. Wisc., Madison (1976*a*).

Weinberger, H. F., "Variational Methods for Eigenvalue Approximation," Society for Industrial and Applied Mathematics, Philadelphia (1972).

Manuscript received June 30, 1977; revision received January 23, and accepted February 22, 1978.

Phase-Plane Analysis of Feedback Control of Unstable Steady States in a Biological Reactor

DAVID DIBIASIO
HENRY C. LIM
WILLIAM A. WEIGAND
and
GEORGE T. TSAO

School of Chemical Engineering
Purdue University
West Lafayette, Indiana 47907

Feedback control of an open loop, unstable, continuous flow, stirred-tank biological reactor was investigated theoretically using several microbial growth models. The closed loop system can be made globally stable, but constraints on the manipulated variable can lead to as many as five steady states, which are either saddle points or locally stable nodes.

SCOPE

Under certain conditions, multiple steady states are possible for continuous cultivation of a pure culture (Yano and Koga, 1969). In such cases, one of the steady states is unstable. When modeling the growth kinetics of microorganisms or designing microbial reactors, it is necessary to specify the specific growth rate of the organism as a function of substrate concentration over the range of possible concentrations. To obtain this data from continuous culture, one must attain operation at unstable steady states. In practical applications (that is, production of single-cell protein, SCP), it may be desirable to operate at or near the maximum specific growth rate for optimal cell production. Such an operation would require knowledge of the system behavior at unstable steady states. It might also be conceivable that an SCP process, with a series of reactors, could be optimized by operating one or more reactors at an unstable steady state. The most expedient means of operation at an unstable steady state is through the application of conventional feedback control. Thus, the study of unstable steady states in the CSTBR plays an important role in the research and design of microbial reactors.

The objective of this paper is to present the results of a theoretical study of the control of unstable steady states in a CSTBR. A number of kinetic models are used in examining the existence of multiple states in the open loop process and the operation at unstable steady states in the closed loop system. The possibility of multiple steady states in the closed loop system will be examined.

The models examined are a classical substrate inhibition model (Yano and Koga, 1969; Edwards et al., 1972), a variable yield model (Chen et al., 1976), a first-order lag model adopted from Young et al. (1970), and a wall growth model (Howell et al., 1972).

Edwards et al. (1972) presented a theoretical study of the control of stable steady states in a CSTBR. Phase-plane analysis was used to show that feedback control could eliminate the instabilities in the system. In a more recent experimental study, Ko and Edwards (1975) demonstrated

Correspondence concerning this paper should be addressed to Henry C. Lim.

that the dynamics of a continuous culture, when represented in a phase space, could be qualitatively predicted by a growth model based on substrate concentration only. Chang and Schmitz (1975) presented the first experimental study of control of an exothermic chemical reaction in a CSTR at an unstable steady state. The current work examines isothermal biological systems and their similarities to the CSTR.

CONCLUSIONS AND SIGNIFICANCE

For any of the growth models used, operation at an unstable state in a CSTBR can be achieved by either proportional or proportional-integral control with realistic controller settings. The systems are globally stable unless constraints are placed on the manipulated variable. Such constraints, which are often unavoidable in practice, can result in a multiplicity of steady states, even though sufficient control action is present.

A comparison of the constant yield vs. variable yield model showed that the stability of the open loop system is a function of the initial substrate concentration. Such information could be useful in the evaluation of a constant yield assumption in experimental work. The variable yield model was shown to be more stable in closed loop operation.

Though an isothermal system, the closed loop behavior in a CSTBR controlled at an unstable steady state, regardless of the model used, is quite similar to that of an exothermic chemical reaction in a CSTR. Reaction paths may show some differences, but regions of stability and the appearance and type of multiple steady states under closed loop conditions are quite similar. The implications for start-up and stable operation of both types of reactors are the same.

Neither operation at an unstable steady state nor the similarities to the CSTR have been investigated previously in a microbial reactor. Operation at unstable states in continuous culture would provide the growth kinetics necessary for design and operation of CSTBR and not rely upon the batch data which do not necessarily represent the continuous reactor situation. This theoretical investigation provides an important precursor for experimental work.

PERTINENT EQUATIONS AND NUMERICAL METHODS

Chemostat Model

The unsteady state cell and substrate balances for the ideal constant-volume reactor with no cells in the feed are

$$dX/dt = X(\mu_x - D) \tag{1}$$

$$dS/dt = D(S_o - S) - \frac{X}{Y}\mu_x \tag{2}$$

where

$$D = F/V, \quad \mu_x = \frac{r_x}{X}, \quad Y = \frac{r_x}{-r_s} \tag{3}$$

The specific growth rate μ_x is typically a function of substrate concentration and for substrate inhibition has historically been used in the form

$$\mu_x = \frac{\mu_m}{K_s/S + 1 + \sum_{j=1}^{n} (S/K_j)^j} \tag{4}$$

The yield Y can be constant or variable (typically a function of S). At steady state, the results for constant yield are simply

$$\mu_x = D \tag{5}$$

$$X_s = Y(S_o - S_s) \tag{6}$$

Since Equation (4) exhibits a maximum, two steady states exist for a fixed value of the dilution rate. The high conversion steady state is stable, while the low conversion steady state is unstable (Yano and Koga, 1969).

The control equations used were those of traditional P and PI control:

$$\text{P:} \quad D = D_s[1 + K_c\epsilon] \quad D \geq 0 \tag{7}$$

$$\text{PI:} \quad D = D_s\left[1 + K_c\epsilon + \frac{K_c}{\tau_I}\int_0^t \epsilon\, dt\right] \quad D \geq 0 \tag{8}$$

where $\epsilon = (X - X_s)/X_s$.

The unsteady state balances and the control equations above form the core of the model used in the simulations conducted.

Phase Plane Analysis

The phase plane technique was used to analyze the behavior of the closed and open loop microbial reactors. Equations (1) and (2) can be combined and used to obtain the phase plane Equation (9):

$$\frac{dS}{dX} = \frac{D(S_o - S) - \frac{X}{Y}\mu_x}{X(\mu_x - D)} \tag{9}$$

After substitution of expressions for μ_x and Y, the solutions to Equation (9) can be obtained by numerical integration on a digital computer. In order to retain flexibility for plotting transients, Equations (1) and (2) were actually integrated simultaneously for different initial cell and substrate concentrations and results plotted as S vs. X. The method of integration was a fourth-order Runge-Kutta scheme with a variable step size option.

Stability Analysis

Linear stability analysis was used as was done by previous authors (Yano and Koga, 1969; Perlmutter, 1972). Local stability is determined by examining the signs of the real part of the eigenvalues derived from the characteristic equation of the linearized system. The condition for stability is that the real parts of the eigenvalues be negative. Real eigenvalues of opposite sign result in a saddle point.

RESULTS AND DISCUSSION

The presentation of the important results of this study is organized into five parts. Each section represents the particular growth model upon which the analysis was made. In every case, a different aspect of the control or nature of the unstable steady state will be illustrated. Two

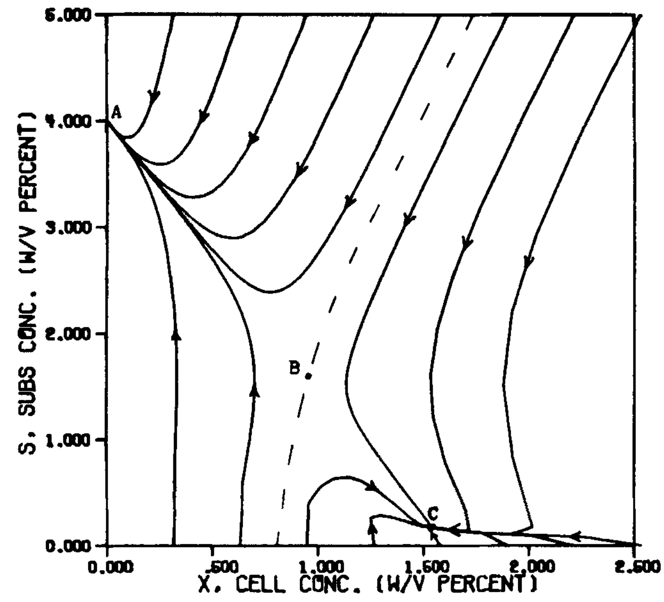

Fig. 1. Phase plane for open loop system, reproduced per Edwards.

control schemes are immediately apparent. When cell concentration is used as the measured variable, and the dilution rate is manipulated to maintain a constant growth rate, the turbidostat results. The nutristat results when substrate concentration is substituted as the measured variable.

Cell concentration was used as the controlled variable in all cases since it is easily obtained as optical density by photometric methods. Though from a control viewpoint the nutristat may in some cases have advantages over the turbidostat (Edwards et al., 1972), from an experimental standpoint its disadvantages are nearly prohibitive in many cases. Therefore, to be able to adequately compare theory with future experiments, the turbidostat was chosen.

Classical Substrate Inhibition Model

The model used was that described in the work of Edwards et al. (1972):

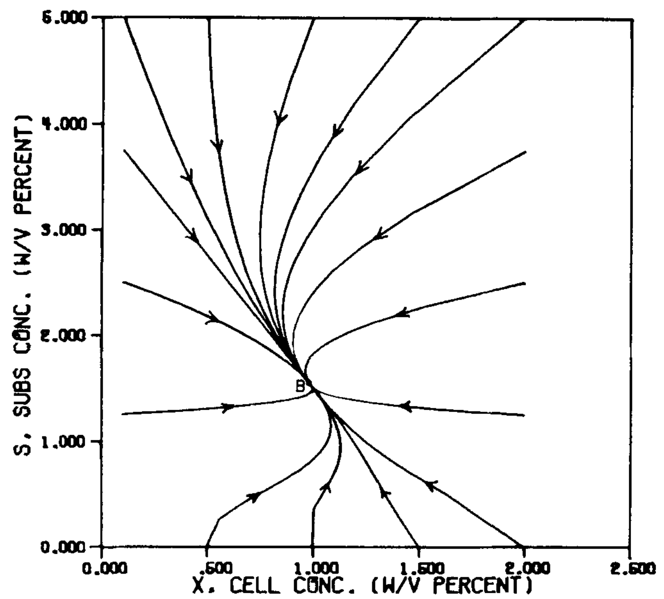

Fig. 2. Phase plane for closed loop system controlled at an unstable steady state using growth model of Edwards (K_c = 1.0).

$$\mu_x = \mu_m S/(K_s + S + S^2/K_i) \qquad (10)$$

The parameter values used were those of Edwards (μ_m = 0.53 l/hr, K_s = 0.12% w/v, K_i = 2.2% w/v, D_s = 0.30 l/hr, Y = 0.4), and feed substrate concentration was 4.0 w/%v.

The open loop phase plane is reproduced per Edwards in Figure 1. The multiplicity of steady states is clearly shown. Point A represents the washout condition. Point C represents the stable steady state and appears in the phase plane as a stable node. Point B represents the unstable steady state and is seen to be a saddle point. The separatrix is indicated by the dashed line. Any initial conditions to the left of it will result in washout, while any initial condition to the right will stabilize at point C.

Edwards et al. demonstrated that proportional control was sufficient to eliminate washout and result in global stability at the open loop stable steady state, point C. Such a phase plane would exist for any positive value of the controller gain.

In this work, the analysis was extended and applied to the unstable steady state. Using the same control equation and gain, the curves shown in Figure 2 resulted. The point B is the saddle point of Figure 1 now converted to a stable node by the control action. This point is now stable for all disturbances in X as the control action forces the system to the singular point and maintains it there.

This clearly illustrates the global stabilization of an unstable steady state using P control only. A smooth nonoscillatory approach to the final state is observed, but there is some cell concentration overshoot. The stability analysis indicates that more than just a positive value of the gain is necessary to stabilize a saddle point. The minimum value depends on the final substrate concentration desired. This will be investigated further in the next section. In order to use physically meaningful controller settings, a calculation estimating the settings for a CSTBR was made using experimental data (DiBiasio, 1977).

The closed and open loop (unstable case) phase planes were found to be very similar to those presented by Chang and Schmitz (1975) for the exothermic CSTR. The unstable steady state was that of the intermediate conversion case, the stable steady state that of the high conversion, and washout the low conversion steady state. In both systems, the stable steady state is a stable node while the unstable steady state is a saddle point. Although the CSTBR is an isothermal system with equations quite different from the nonisothermal CSTR, the behavior in the phase space appeared quite similar. This is evident in the relative location and type of steady states and the qualitative appearance of the reaction paths.

The effects of constraints on the manipulated variable were investigated. If the upper limit is low, a plot such as Figure 3 results. Here, the steady state dilution rate is 0.3 l/hr, and the upper limit is 0.35 l/hr. The stabilized steady state is no longer unique, as it is evident that another steady state is possible. Point B remains the same as in Figure 2 but is now only locally stable. The constraint on D has resulted in the appearance of a stable steady state designated by C′. The saddle point B′ defines a new separatrix. B′ and C′ are the unstable and stable steady states, respectively, that correspond to the open loop system with D = 0.35 l/hr. In other words, the controller is saturated at the upper limit in this region, and the system behaves for the most part as if there is no control. Start-up of such a system, with the intent of reaching point B, would not be difficult as the majority of the initial conditions fall to the left of the separatrix. This plot is again analogous to the exothermic CSTR for the high residence time case. The result of increased residence time in the CSTR was to cause controller saturation at the high limit of heat removal. The desired stabilized saddle point was no longer unique, as a new steady state corresponding to maximum heat removal was possible.

The result, with the addition of a lower limit on the manipulated variable, is given in Figure 4. The limit was 0.25 l/hr. The system now has three stable steady states, and these are delineated by two separatrices. Initial values to left of the separatrix No. 1 will result in washout to the feed condition, point A. This separatrix passes through a saddle point given by A′ which is the unstable steady state for the open loop system with $D = 0.25$ l/hr. The controller is thus saturated at the lower limit of the manipulated variable. Separatrix No. 2 and points B′ and C′ are identical to those in Figure 3. The desired final state is again point B. However, initial conditions to the right of separatrix No. 1 and left of separatrix No. 2 must be used if such a state is to be reached. There is only a small region around the stabilized saddle point B where the controller is not saturated.

The start-up of a system, such as that shown in Figure 4, would indeed be difficult if point B were desired. One needs to prevent the system from moving to washout or an unwanted steady state. This is evident by comparing the small region of asymptotic stability for point B in Figure 4 to that of Figure 3.

The left half of Figure 4 is now similar to the adiabatic CSTR. In the system of Chang and Schmitz, constraints on the manipulated variable and the system parameters were such that five steady states were not possible in one phase space. Also, with the given limits, a residence time existed such that global stabilization could always be achieved. In our case, the only variable left to change that could eliminate the saddle points in Figure 4 is S_o. All other conditions constant, a reduction to 2.0 w/v% is sufficient to partially accomplish this. The washout state can be eliminated, but unless one restricts the maximum initial cell concentrations, the instability caused by the controller saturation at the upper limit will always exist. The effect of lowering S_o is to decrease the controller error for low initial X. Hence, the dilution rate prescribed by the controller is never saturated at the lower limit long enough to cause washout. The effect is detrimental of course for high initial X. The whole result could be reversed by increasing S_o. This would eliminate saturation and hence instability at the upper limit but would retain the possibility of washout. In theory, however, the constraints on the dilution rate could always be made small enough to result in the five steady state phase planes.

Consider a situation where one is given the upper and lower limits on the dilution rate and desired residual substrate concentration corresponding to an unstable steady state within these limits. The expedient method of avoiding a system with a phase plane like Figure 4 would be to lower S_o. This is more easily accomplished (that is, by dilution) than increasing S_o which would require concentrating the feed stream. The desired stabilized saddle point would necessarily have a lower steady state cell concentration but would be more easily reached as any low value of X would guarantee the desired final steady state. This would, in addition, reduce inoculum requirements or initial time spent in batch operation prior to start-up.

Using a standard substrate inhibition model control of an unstable steady state is thus readily achieved. Constrained manipulated variables can result in as many as five possible steady states in the same phase space. Although the system is isothermal and quite different than an exothermic CSTR, surprisingly similar behavior is obtained. Such behavior has not been shown before for a CSTBR.

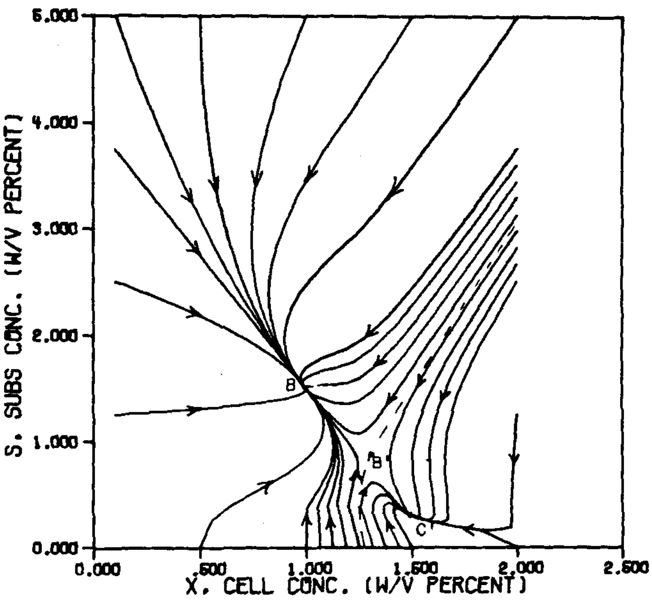

Fig. 3. Phase plane for control at an unstable steady state with an upper limit on D (Edwards' model, $K_c = 1.0$, $0 \leq D \leq 0.35$ l/hr).

Variable Yield Model

This model as proposed by Chen et al. (1976), based on batch data of a methanol utilizer, was used in the form

$$\mu_x = \left[\frac{aS}{b+S}\right]\left[\frac{1-cS}{1-dS}\right] \tag{11}$$

The variable yield function is

$$Y(S) = Y_o\left[\frac{1-cS}{1-dS}\right] \tag{12}$$

Values of the parameters and steady state conditions are summarized in Table 1.

The local stability criteria for this model for the closed loop case with P control is given by

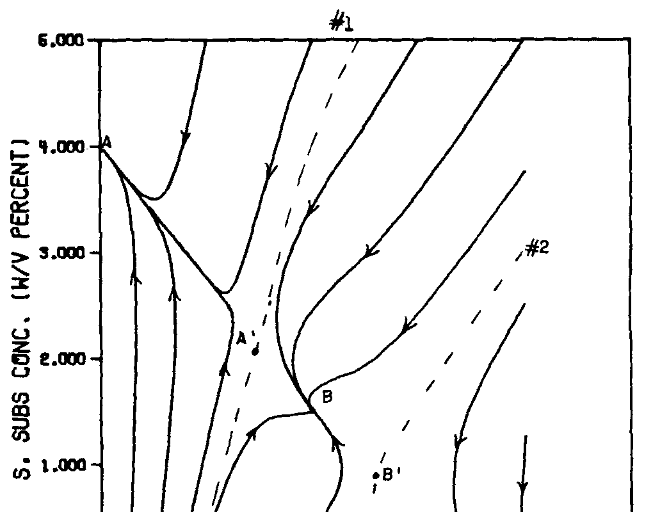

Fig. 4. Phase plane for control at an unstable steady state with upper and lower limits on D (Edwards' model, $K_c = 1.0$, $0.25 \leq D \leq 0.35$ l/hr).

Table 1. Parameters and Steady State Values for Variable Yield Model

Parameter	Value
a	0.455 l/hr
b	0.00712 w/v%
c	0.216 l/(w/v%)
d	0.0598 l/(w/v%)
Y_o	0.6
Steady state	
D	0.35 l/hr
S_o	2.00 w/v%
Unstable, X	0.37 w/v%
S	1.23 w/v%
Stable, X	0.027 w/v%
S	1.18 w/v%

Table 2. Steady State Values and Constants for Open Loop Comparison of Constant and Variable Yield Models

Parameter	Value
Unstable, X	0.10 w/v%
S	3.82 w/v%
D_s	0.10 l/hr
Y_c	0.15
Y_o	0.6
S_o	4.5 w/v%

$$\left[(S_o - S_s) - \frac{\gamma}{K_c Y_o} \right] D_s \Phi < \frac{abX_s}{(b + S_s)^2 Y_o} + D_s \quad (13)$$

where

$$\gamma = \frac{aS_s}{b + S_s} \quad (14)$$

$$\Phi = \left[\frac{a - 2eS_s}{aS_s - eS_s^2} \right] - \left[\frac{f - 2dS_s}{b + fS_s - dS_s^2} \right] \quad (15)$$

and

$$e = ac, \quad f = 1 - bd \quad (16)$$

It can be shown that $D_s\Phi$ is just the slope of the growth curve at the steady state. For unstable steady states, this is always negative. Thus, Equation (13) illustrates the fact that a minimum gain must be used to insure stabilization and control of an otherwise unstable steady state. This is contrary to the control of a stable steady state, where any positive value of K_c will suffice for control.

The open loop phase plane is not reproduced here. It is analogous to Figure 1, and the control of the stable steady state is similar to the previous work of Edwards and needs no further explanation. Interesting features of the open loop equations are presented later.

The saddle point of this system can be converted to a stable node by the application of proportional control. The phase plane is similar to Figure 2 and is not presented. A gain of 1.0 was sufficient to realize global stabilization. By applying suitable constraints to the manipulated variable, the system will conform to behavior equivalent to that in Figures 3 and 4.

The effect of adding integral control was then studied. PI control offers the advantage of eliminating offset but sacrifices ultimate time to steady state. The unstable steady state was easily stabilized for different degrees of integral action. Varying the relative amounts of P and PI control action changed the shape of the phase plane trajectories significantly, though global stabilization was always realized.

The application of constraints on the manipulated variable again results in a multiplicity of steady states with PI control. Using $0.325 \leq D \leq 0.4$ l/hr, a phase plane analogous to Figure 4 results. The same upper and lower limits were used as for the case of P control only. Thus, the appearance of multiple steady states is not dependent on the type of control but rather on the degree of control. It is also possible to have multiple steady states without having any limit on the manipulated variable but by having little control action (with a small gain). In the cases illustrated, multiple steady states were obtained in systems that were originally globally stable by using identical control constants and the described constraints. Given the limits on the manipulated variable in such cases, the control action could be reduced by reducing the gain and reset to a point where saturation never occurs long enough to stabilize at an unwanted steady state, yet the system is still globally stable at the desired saddle point. This would, however, sacrifice some degree of the desired transient response.

Variable vs. Constant Yield

The effect of variable yield in the neighborhood of the unstable steady state in both open and closed loop situation was investigated. The S value is the same for each case, but to obtain the X value, a much lower yield is needed in the constant yield case. The values are summarized in Table 2. The comparison is then between the system represented by Equation (1) for X and S by Equation (17) for constant yield and Equation (18) for variable yield:

$$dS/dt = D(S_o - S) - X \left[\frac{1}{Y_c} \left(\frac{aS}{b + S} \right) \left(\frac{1 - cS}{1 - dS} \right) \right] \quad (17)$$

$$dS/dt = D(S_o - S) - X \left[\frac{1}{Y_o} \left(\frac{aS}{b + S} \right) \right] \quad (18)$$

In examining which model describes a more stable system, it is first necessary to refer to a typical open loop phase plane (that is, Figure 1). One can see that for any given initial S, the initial value of X is the sole determining factor in reaching a stable steady state. The larger the region of asymptotic stability, that is, the further the separatrix lies to the left of the phase plane, the more stable the system is.

Since, for given initial X and S values, the equation for dX/dt is identical for both models, it is necessary to examine the equations for dS/dt, Equations (17) and (18), for the purpose of comparing stabilities. Realizing that the term in brackets in Equations (17) and (18) is the specific substrate consumption rate, they can be rearranged to the form

$$X = \frac{dS/dt - D(S_o - S)}{\mu_s} \quad (19)$$

It is then postulated that the numerator, for identical initial S values, is roughly the same for both models. This is not strictly true but can be used to obtain qualitative initial estimates of system behavior. Thus, the model which gives the highest value of μ_s would be more stable. Qualitatively, this can be obtained by directly comparing the μ_s for each model. The relationship is plotted in Figure 5. The cell growth rate is the same for both cases. The two substrate consumption curves necessarily intersect at the unstable steady state. At initial substrate concentrations greater than the unstable steady state (3.82 w/v%), one can see from Figure 5 that the substrate consumption rate is always greater for the variable yield case, and this would be the more stable system. For initial values of S lower than the unstable steady state, the substrate con-

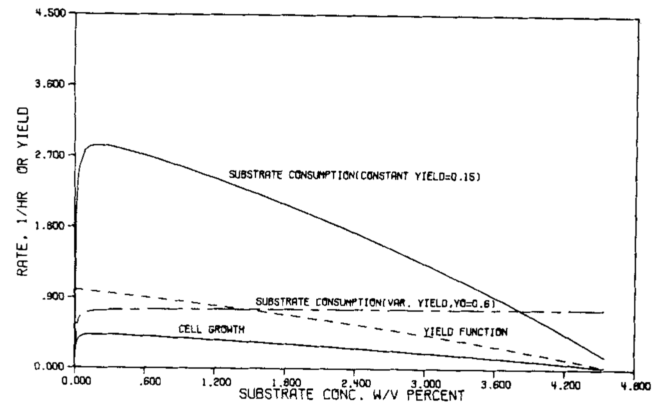

Fig. 5. Comparison of substrate consumption rates for variable and constant yield models with the same growth rate function.

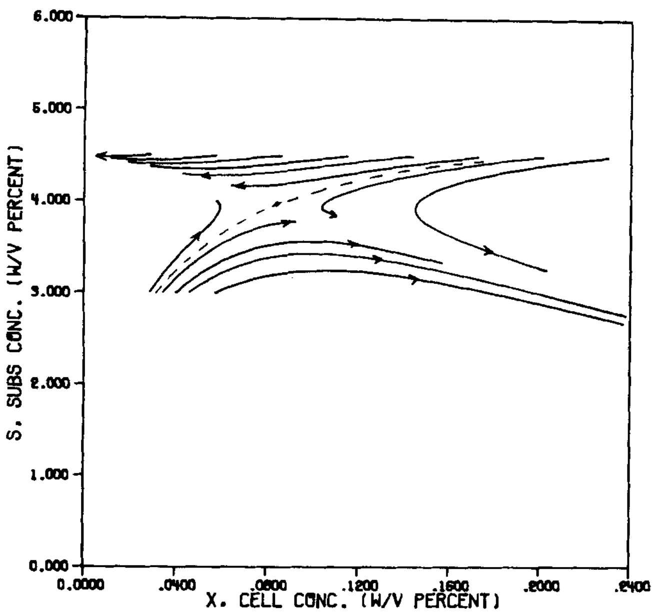

Fig. 6. Open loop phase plane for constant yield system. (Model of Chen, parameters in Table 2.)

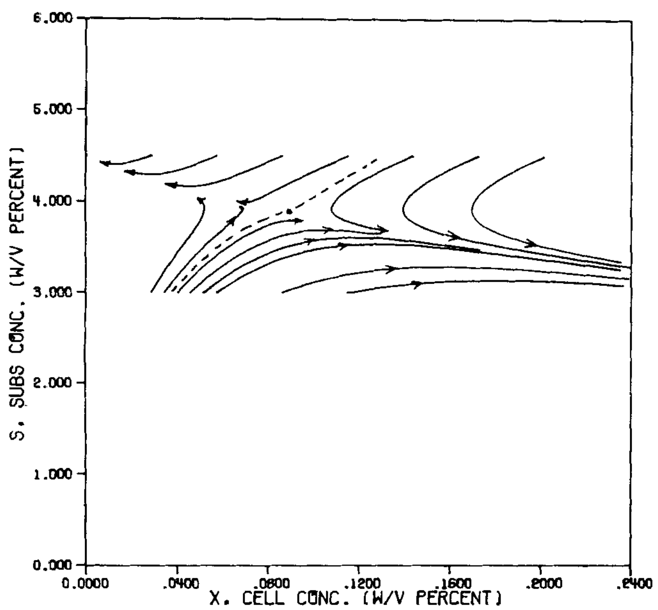

Fig. 7. Open loop phase plane for variable yield system. (Parameters in Table 2.)

sumption rate for the constant yield case is always greater, and this would be the more stable system.

The open loop phase planes around the saddle point for identical initial conditions are plotted for constant yield in Figure 6 and variable yield in Figure 7. These numerical results confirm the above argument. In both figures, paths to the left of the separatrix approach washout and those on the right approach the stable steady state. It is seen that for each case the separatrix intersects the horizontal line representing the initial substrate concentration at different points. By visual inspection, then, one can determine that the variable yield system is more stable for high initial S and less stable for low initial S. In general, the total region of asymptotic stability in this case is greatest for the variable yield model.

Such an analysis would be useful in experimental work aimed at modeling microbial growth kinetics. Obviously the transients of each model show quite different behavior. For identical initial conditions, one shows washout while the other predicts a stable steady state for the final state. Comparison of experimentally observed transients near an unstable steady state to those predicted by each model would give a good insight into the validity of an assumption of constant yield.

The above analysis is also useful when applied to the closed loop system to speculate which model would require the lowest gain (P control only). The results showed that washout began to occur from initial substrate concentrations greater than the unstable steady state, and thus the variable yield model required less gain to retain global stability (DiBiasio, 1977). This was in agreement with the prediction.

For the variable yield case, a comparison of the minimum gain obtained by numerical integration and that calculated by analytical approximation [Equation (13)] showed quite good agreement.

The variable yield model requires less gain to maintain an unstable steady state once it is reached. Referring again to Figure 5, one can see that a positive disturbance from the steady state would tend to washout more slowly for the variable yield case as the μ_s is greater. On the other hand, a negative disturbance in S would tend to the stable state more slowly for variable yield as the μ_s is now lower. Thus, a lower gain would be required for variable yield to maintain stability at a saddle point once it is reached, and

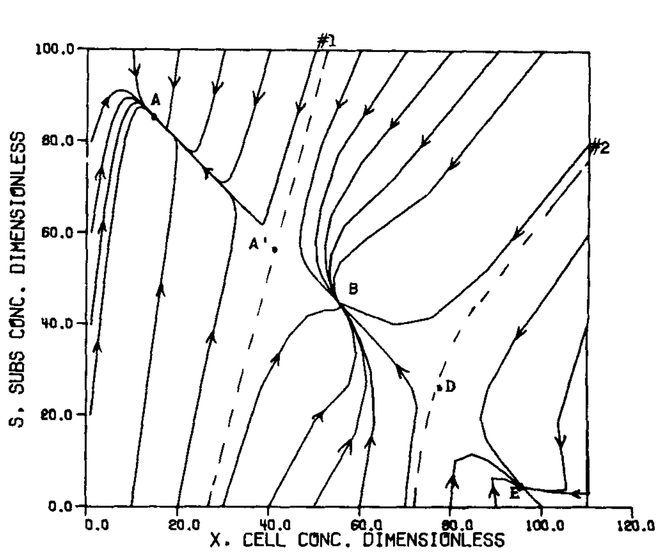

Fig. 8. Phase plane for control at an unstable steady state using wall growth model and upper and lower limits on the dilution rate (*P* control, $K_c = 0.35, 0.26 \leq D \leq 0.33$ l/hr).

the system would be less sensitive to imposed perturbations.

First-Order Lag Model

The model of Young et al. (1970) previously discussed does not directly apply to the case of substrate inhibition. However, a similar analysis can be made using their basic premise. A crude model was derived in which the overall transport process of substrate across the cell membrane was approximated by a first-order lag:

$$\tau \frac{d\overline{S}_i}{dt} + \overline{S}_i = K\overline{S}_x \tag{20}$$

Growth rate was described as a function of internal cell substrate concentration [Equation (10) with $S = S_i$].

Numerical results were obtained for one case using the growth constants from Edwards data with $K = 1.0$ and $\tau = 0.15$ hr. The open loop results are quite similar to those previously discussed, except that damped oscillations in substrate concentration appeared for some initial conditions in approaching the stable steady state. The other models do not predict such behavior. As before, stabilization of the desired unstable steady state can be accomplished with proportional control. Most of the oscillation is eliminated in the closed loop system which is globally stable at the desired singular point. Application of constraints on the dilution rate resulted in similar multiplicities as discussed earlier and is not repeated here for this model.

Wall Growth Model

Howell et al. (1972) presented the mass balances for the wall growth model and the open loop phase plane and stated that the presence of wall growth eliminates washout and results in a low conversion stable steady state corresponding to substrate consumption due to wall growth only. A saddle point, representing the unstable steady state, and a stable node, representing the high conversion stable steady state, also exist. They did not carry out the closed loop analysis.

With proportional control, and a gain of 0.35, the phase plane was globally stabilized at the unstable steady state, and no unusual behavior was apparent.

Constraints were then placed on the manipulated variable. The steady state value of D was 0.28 l/hr, and the limits were 0.33 and 0.26 l/hr. Figure 8 shows that five steady states exist and that the stabilized saddle point is no longer unique. Point B remains the desired stabilized saddle point of the original open loop system. Its region of asymptotic stability is now severely restricted. The description is similar to that for Figure 4, save for the washout state. The low conversion steady state, point A, is the same as the open loop system. This is necessarily so, since the lower limit on the dilution rate was sufficient to wash out all suspended cells, and therefore what remains is just the constant wall growth concentration.

A similar plot to Figure 8, with two saddle points and three stable nodes, could be obtained for a CSTR similar to that of Chang and Schmitz if the manipulated variable constraints were tightened sufficiently. Although the shape of the reaction paths to these states would differ somewhat, the qualitative similarity between CSTR and the CSTBR is again evident.

The comparison of the microbial reactor and the exothermic CSTR has been limited strictly to a qualitative discussion of the number and types of steady states existing in the phase space. No general conclusions have been drawn. The analogy does, however, raise interesting questions as to whether other known behavior in the chemical reactor (stable and unstable foci and limit cycles) can be observed in a biological reactor and whether more use than just a passing notice can be made of the comparison.

Experimental work currently is being conducted, using methanol utilizing organisms, to obtain evidence for and extensions of the theory described in this paper.

ACKNOWLEDGMENT

This work was supported in part by a grant from the National Science Foundation, ENG-75-17796.

NOTATION

- a = growth model constant, l/time
- a_{ij} = constants in linear model
- b = growth model constant, conc.
- c = growth model constant, conc.
- CSTR = continuously fed stirred-tank reactor
- CSTBR = continuously fed stirred-tank biological reactor
- d = growth model constant
- D = dilution rate, l/time
- F = volumetric flow rate
- K_c = proportional controller gain, dimensionless
- K = constant in first-order lag model
- K_s = growth rate constant, conc.
- K_i = inhibition constant, conc.
- K_j = inhibition constant, conc.
- r_s = substrate consumption rate, conc./time
- r_x = cell growth rate, conc./time
- $\overline{S}$ = deviation variable of substrate concentration
- S = substrate concentration in reactor (external to cells)
- S_o = feed substrate concentration
- S_i = internal cell substrate concentration
- S_x = external cell substrate concentration
- SCP = single-cell protein
- t = time
- V = reactor volume
- X = cell concentration
- x = deviation variable of cell concentration, $X - \alpha$
- y = deviation variable of substrate concentration, $S - \beta$
- Y_o = overall yield coefficient
- Y, Y_c = constant yield coefficient
- $Y(S)$ = variable yield function

Greek Letters

- γ = parameter defined by Equation (14)
- ϵ = controller error
- λ = eigenvalue
- μ_m = maximum specific growth rate, l/time
- μ_x = specific growth rate of cells, l/time
- μ_s = specific rate of substrate consumption, l/time
- τ_I = integral time constant, time
- τ = constant in first-order lag model, time
- Φ = parameter defined by Equation (15)

Subscripts

- i = internal substrate concentrations
- j = integer constant
- s = steady state value
- x = external substrate concentrations

Superscripts

- j = integer constant
- — = deviation variable

LITERATURE CITED

Chang, M., and R. A. Schmitz, "An Experimental Study of Oscillatory States in a Stirred Reactor," *Chem. Eng. Sci.*, **30**, 21 (1975).

———, "Feedback Control of Unstable States in a Laboratory Reactor," *ibid.*, 837 (1975).
Chen, B. J., H. C. Lim, and G. T. Tsao, "A Model for Bacterial Growth on Methanol," *Biotechnol. Bioeng.*, 18, 1629 (1976).
DiBiasio, D., M.S. thesis, School of Chemical Engineering, Purdue Univ., W. Lafayette, Ind., (1977).
Edwards, V. H., R. C. Ko, and S. A. Balogh, "Dynamics and Control of Continuous Microbial Propagators Subject to Substrate Inhibition," *Biotechnol. Bioeng.*, 14, 939 (1972).
Howell, J. A., C. T. Chi, and U. Pawlowsky, "Effect of Wall Growth on Scale-Up Problems and Dynamic Operating Characteristics of the Biological Reactor," *ibid.*, 253 (1972).
Ko, R. C., and V. H. Edwards, "Transient Kinetics of Yeast Growth in Batch and Continuous Culture with an Inhibitory Carbon and Energy Source," *ibid.*, 17, 65 (1975).
Perlmutter, D. D., *Stability of Chemical Reactors*, Prentice-Hall, Englewood Cliffs, N. J. (1972).
Yano, T., and S. Koga, "Dynamic Behavior of the Chemostat Subject to Substrate Inhibition," *Biotechnol. Bioeng.*, 11, 139 (1969).
Young, T. B., D. F. Bruley, and H. R. Bungay, III, "A Dynamic Mathematical Model of the Chemostat," *ibid.*, 12, 747 (1970).

Manuscript received August 5, 1977; revision received February 20, and accepted March 6, 1978.

Dynamics and Control of the Activated Sludge Wastewater Process

The dynamics of the activated sludge process are governed under certain conditions by interactions between the reactor and settler through sludge recycle. Settler underloading can lead to extremely sluggish system response, indicating that effective sludge height regulation is an important control objective. This objective may be in conflict with the need to maintain small variations in reactor solids concentration. An effective compromise can be achieved by using ratio control on both sludge recycle and settler underflow. This control policy does not require sludge storage.

URI ATTIR
and
MORTON M. DENN
Department of Chemical Engineering
University of Delaware
Newark, Delaware 19711

SCOPE

The activated sludge process is the most widespread method of secondary wastewater treatment. The process consists of a continuous flow biochemical reactor followed by a continuous settler, with recycle of active biological sludge to the reactor. The process typically experiences large changes in feed flow rate and dissolved organics over time scales that are comparable both to reactor and settler residence times and to the reciprocal of the pseudo first-order rate constant for reaction of organics. Thus, the process will rarely operate at steady state and may require controls for good performance.

The settler response can be important in determining the overall process dynamics because of the close coupling between reactor and settler. Previous control studies have used inadequate models of the settler that cannot account for possible adverse interactions between the units and resulting deterioration of performance. This paper describes the results of a simulation study of the dynamics and control of the activated sludge process, using a recently developed model of the dynamics of continuous sedimentation that accounts for several distinct modes of operation. Interactions between reactor and settler dynamics are particularly considered.

CONCLUSIONS AND SIGNIFICANCE

The simulated dynamical response deteriorates significantly when the settler is underloaded, because the buffering action of the sludge blanket is lost and reactor disturbances are positively reinforced by the recycle loop. This suggests that a primary goal of a control system should be to maintain the sludge level in the settler. Sludge blanket control can lead to large swings in reactor solids concentration, however, which is also undesirable. Conversely, a control system designed to regulate only the reactor can cause the settler to become underloaded or overloaded.

An effective compromise between these two conflicting goals can be achieved using an elementary strategy of hydraulic control, in which the rates of settler underflow and recycle to the reactor are kept at constant ratios to the feed flow rate. This strategy diminishes variations in both reactor solids concentration and settler sludge blanket height while requiring only flow measurements. Unlike some other proposed control strategies, no storage of sludge is required.

Uri Attir is with the Diamond Shamrock Company, Painesville, Ohio 44077.
0001-1541-78-1537-0693-$00.85.